RMSC 4003
Statistical Modeling in Financial Markets
Tutorial 3 Solution

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1 Portfolio Variance and Feasible Sets

1.1 Portfolio Variance

Example 1.1 (Portfolio Variance)**.** Let $r_P = \sum_{i=1}^n w_i r_i$. Show that $\mathrm{Var}(r_P) = \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}$.

Proof.

$$\begin{aligned}
\mathrm{Var}(r_P) &= \mathrm{E}(\sum_{i=1}^n w_i r_i - \sum_{i=1}^n w_i \mu_i)^2 \\
&= \mathrm{E}\left(\sum_{i=1}^n w_i (r_i - \mu_i) \sum_{j=1}^n w_j (r_j - \mu_j)\right) \\
&= \mathrm{E}\left(\sum_{i=1}^n \sum_{j=1}^n w_i w_j (r_i - \mu_i)(r_j - \mu_j)\right) \\
&= \sum_{i=1}^n \sum_{j=1}^n w_i w_j \sigma_{ij}.
\end{aligned}$$

□

1.2 Feasible Sets

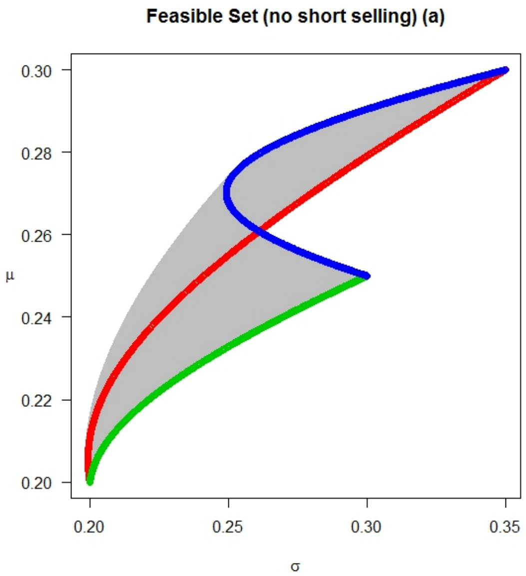
Suppose there are 3 assets and short selling is not allowed. Situation (a):

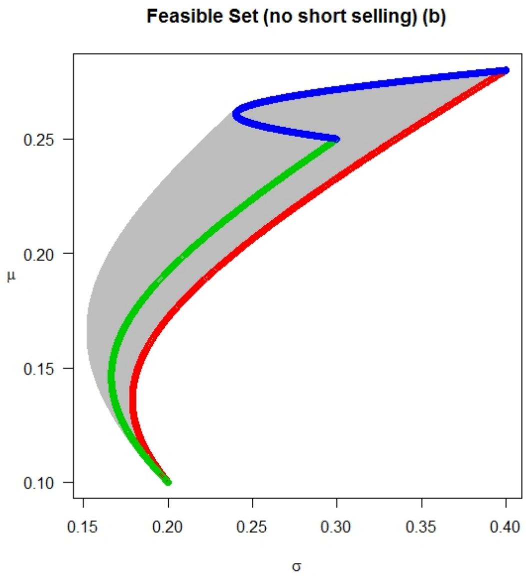
$$\mu = \begin{pmatrix} 0.2 \\ 0.25 \\ 0.3 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.04 & 0.042 & 0.035 \\ 0.042 & 0.09 & 0.021 \\ 0.035 & 0.021 & 0.1225 \end{pmatrix}.$$

Situation (b):

$$\mu = \begin{pmatrix} 0.1 \\ 0.25 \\ 0.28 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 0.04 & 0 & 0 \\ 0 & 0.09 & 0 \\ 0 & 0 & 0.16 \end{pmatrix}.$$

The following graphs are obtained:





2 To find the minimum variance point

To find the minimum variance point, we are solving the following problem:

$$\text{minimize } \frac{1}{2}\sum_{i,j=1}^{n} w_i w_j \sigma_{ij}$$

$$\text{subject to } \sum_{i=1}^{n} w_i = 1.$$

Similar to what we did in Markowitz Model, we use the Lagrange multipliers methods. That is, we solve

$$\begin{aligned}\sum_{j=1}^{n} w_j\sigma_{ij} - \lambda &= 0, \quad i = 1,\ldots,n\\ \sum_{i=1}^{n} w_i &= 1.\end{aligned}$$

Remark 2.1. In solving the Markowitz model, we do allow short selling (mathematically, there is no constraint $w_i \geq 0$). If we want to solve the problem when short selling is not allowed. One is solving an optimization problem with equality constraint and inequality constraint (see Karush-Kuhn-Tucker conditions, which is out of our scope). One can also solve the optimization numerically.

Example 2.1 (Solving Markowitz Model and minimum variance point with R)**.** Consider a three-asset universe with annualized returns given by r_1, r_2 and r_3, respectively. Suppose that the expected value and the covariance matrix of the returns are

$$\mu = \begin{pmatrix} 0.15 \\ 0.10 \\ 0.10 \end{pmatrix} \qquad \Sigma = \begin{pmatrix} 3 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

(a) Using the "solve" function in R, find the general solutions of the minimum-variance portfolio for a targeted expected portfolio return μ.

(b) Find the minimum-variance portfolio with expected return $\mu = 15\%$. What is the corresponding portfolio variance?

(c) Find the (global) minimum-variance point, and its portfolio variance. Hint: use the fact that w_1 and w_3 play a symmetric role in the variance of the portfolio.

Remark 2.2. The unit (percentage or decimal) for the variance-covariance matrix will not affect the optimization.

Solution. (a) R code:

```
Sigma <-matrix(c(3,1,0,1,3,1,0,1,3),ncol=3)
mu <-c(0.15,0.1,0.1)
A<-cbind(Sigma,mu,c(1,1,1))
A<-rbind(A,c(mu,0,0),c(1,1,1,0,0))
round(solve(A),4)
```

Using R, the solutions for the minimum-variance portfolio for a given portfolio mean is

$$\begin{aligned}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ -\lambda_1 \\ -\lambda_2 \end{pmatrix} &= \begin{pmatrix} 3 & 1 & 0 & 0.15 & 1 \\ 1 & 3 & 1 & 0.10 & 1 \\ 0 & 1 & 3 & 0.10 & 1 \\ 0.15 & 0.10 & 0.10 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 20 & -2 \\ 0 & 0.25 & -0.25 & -15 & 2 \\ 0 & -0.25 & 0.25 & -5 & 1 \\ 20 & -15 & -5 & -1500 & 180 \\ -2 & 2 & 1 & 180 & -23 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \\ \mu \\ 1 \end{pmatrix} = \begin{pmatrix} 20\mu - 2 \\ -15\mu + 2 \\ -5\mu + 1 \\ -1500\mu + 180 \\ 180\mu - 23 \end{pmatrix}\end{aligned}$$

Therefore, $w_1 = 20\mu - 2$, $w_3 = -15\mu + 2$ and $w_3 = -5\mu + 1$.

(b) Put $\mu = 0.15$, we have $(w_1, w_2, w_3) = (1, -0.25, 0.25)$. Therefore, the minimum variance portfolio is given by $r_P = r_1 - 0.25r_2 + 0.25r_3$, with the corresponding portfolio variance

$$\sigma_P^2 = \sum_{i=1}^{3} w_i^2\sigma_i^2 + 2\sum_{i\neq j} w_i w_j \sigma_{ij} = 3(w_1^2 + w_2^2 + w_3^2) + 2(w_1w_2 + w_2w_3) = 2.75.$$

(c) Let $r_P = w_1r_1 + w_2r_2 + w_3r_3$. Then the portfolio variance is

$$\sigma_P^2(w_1, w_2, w_3) = \sum_{i=1}^{3} w_i^2\sigma_i^2 + 2\sum_{i\neq j} w_i w_j \sigma_{ij} = 3(w_1^2 + w_2^2 + w_3^2) + 2(w_1w_2 + w_2w_3).$$

Note that σ_P^2 is symmetric in w_1 and w_3. Hence σ_P^2 is minimized when $w_1 = w_3$. Therefore,

$$\sigma_P^2 = 3(w_1^2+w_2^2+w_3^2)+2(w_1w_2+w_2w_3) = 3[2w_1^2+(1-2w_1^2)]+4w_1(1-2w_1) = 10(w_1-0.4)^2+1.4.$$

Hence, σ_P^2 is minimized at 1.4 when $w_1 = 0.4$, which gives $w_3 = 0.4$ and $w_2 = 0.2$. That is, the minimum variance portfolio is the portfolio $(w_1, w_2, w_3) = (0.4, 0.2, 0.4)$.

Alternatively, the following R code solves the problem:

```
Sigma <-matrix(c(3,1,0,1,3,1,0,1,3),ncol=3)
A<-cbind(Sigma,c(1,1,1))
A<-rbind(A,c(1,1,1,0))
x = c(0,0,0,1)
solve(A,x)
```

3 Remarks on Homework 2

Question 3 in Homework 2: There are there assets with rates of return r_1, r_2, r_3 such that the mean and covariance matrix are:

$$\mu = \begin{pmatrix} 0.4 \\ 0.8 \\ 0.8 \end{pmatrix} \quad \Gamma = \begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

(a) Find the minimum-variance portfolio. Note that by symmetry, you may take $w_1 = w_3$.

(b) Find another portfolio by setting $\lambda_1 = 1, \lambda_2 = 0$.

(c) Normalize this portfolio.

(d) If the risk-free rate is $\mu_f = 0.2$, find the efficient portfolio of risky assets.

For part (a), you may refer to Example 2.1 in this notes. For part (b) and (c), you are asked to find another portfolio (on the minimum-variance set) by setting $\lambda_1 = 1, \lambda_2 = 0$. The λ's refer to the Lagrange multipliers in the Markowitz model. You may wonder why we can find the portfolio by choosing λ_1 and λ_2 and then normalize the portfolio (weight).
Reason: Suppose you fix λ_1 and λ_2 and solve the following equations for w_j's:

$$\sum_{j=1}^{n} w_j \sigma_{ij} = \lambda_1 \mu_i + \lambda_2, \quad i = 1, \dots, n.$$

Then w_j's will satisfy $\sum_{i=1}^{n} w_i \mu_i = \mu^*$ for some μ^*. In general, $\sum_{i=1}^{n} w_i \neq 1$. To make $\sum_{i=1}^{n} w_i = 1$, we can normalize w_i's by setting

$$v_i := \frac{w_i}{\sum w_i} \quad \text{for all i=1,\dots,n.}$$

Then $v_1, \dots, v_n, \lambda_1^* := \frac{\lambda_1}{\sum w_k}$ and $\lambda_2^* := \frac{\lambda_2}{\sum w_k}$ will satisfy

$$\begin{aligned} \sum_{j=1}^{n} v_j \sigma_{ij} &= \frac{\lambda_1}{\sum w_k} \mu_i + \frac{\lambda_2}{\sum w_k}, \quad i = 1, \dots, n \\ \sum_{j=1}^{n} v_j \mu_j &= \frac{\mu^*}{\sum w_i} \\ \sum_{j=1}^{n} v_j &= 1. \end{aligned}$$

That means $(v_1, \dots, v_n)$ is the minimum-variance portfolio subject to the constraint that

$$\begin{aligned} \sum_{j=1}^{n} v_j \mu_j &= \frac{\mu^*}{\sum w_i} \qquad \text{(the targeted portfolio mean)} \\ \sum_{j=1}^{n} v_j &= 1. \end{aligned}$$

For part (d), you may assume risk-free borrowing is allowed so that what we want to find is the tangency portfolio.

In part (b) and (d), you need to solve a linear system. An useful to solve it by hand is to use the augmented matrix (recall what you learned in linear algebra). See Example 7.1 in this notes.

4 Two-Fund Theorem (when there is no risk-free asset)

Theorem 4.1 (Two-Fund Theorem)**.** *Any efficient fund (or portfolio) can be replicated, in terms of mean and variance, as a combination of two efficient funds. Mathematically, if $\boldsymbol{w^1}$ and $\boldsymbol{w^2}$ are two efficient portfolios, then $\boldsymbol{w}^* := \alpha\boldsymbol{w^1} + (1-\alpha)\boldsymbol{w^2}$ is also an efficient portfolio.*

5 Inclusion of Risk-free Asset

Definition 5.1. *A risk-free asset is an asset with $\sigma = 0$*

However, there will be interest rate risk and reinvestment risk when the maturity of the risk-free asset does not match your investment period/ holding period. There will also be currency risk if your risk-free asset is denominated in other currency. (Recall the major risks in financial markets: interest rate risk, equity risk, currency risk, credit risk, commodity risk, reinvestment risk, inflation risk, etc.)

In general, any combination of a risky portfolio (consisting of stocks, with mean μ and variance σ^2) and a risk-free asset will lie on the straight line connecting the risk-free asset and that portfolio on the $\mu - \sigma$ diagram. Indeed, if $r_p = \alpha r + (1-\alpha)r_f$, then

$$\begin{aligned}\mu_p &= \alpha\mu + (1-\alpha)r_f \\ \sigma_p &= \alpha\sigma.\end{aligned}$$

Eliminating α, we have

$$\mu_p = r_f + (\frac{\mu - r_f}{\sigma})\sigma_p.$$

- When only risk-free lending is allowed, the efficient frontier will consists of a straight line segment from the risk-free asset to the tangency portfolio T and a curved segment from T to the end of the original efficient frontier without the risk-free asset.
- When risk-free borrowing is allowed, the efficient frontier will be the line from the risk-free asset passing through T and beyond.

6 One-Fund Theorem (when there is risk-free asset)

Theorem 6.1 (One-Fund Theorem)**.** *There is a single fund T of risky assets (called the tangency portfolio) such that any efficient portfolio can be constructed as a combination of this fund T and a risk-free asset.*

To determine the tangency portfolio T:

(1) Solve $\Sigma v = b$ where

$$\Sigma = \begin{pmatrix} \sigma_{11} & \dots & \sigma_{1n} \\ \vdots & \ddots & \vdots \\ \sigma_{n1} & \dots & \sigma_{nn} \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}, \quad b = \begin{pmatrix} \mu_1 - r_f \\ \vdots \\ \mu_n - r_f \end{pmatrix}.$$

(2) Normalize v_i by

$$w_i = \frac{v_i}{\sum_{j=1}^n v_j} \quad \forall i = 1, 2, \ldots, n.$$

7 Examples

Example 7.1 (Tangency portfolio; augmented matrix)**.** Suppose that there are three assets with the following mean and variance-covariance matrix:

$$\mu = \begin{pmatrix} 0.13 \\ 0.23 \\ 0.33 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 1 & 2 \\ 1 & 3 & 2 \\ 2 & 2 & 4 \end{pmatrix}$$

What is the tangency portfolio when the risk-free asset rate $r_f = 3\%$.

Solution.

$$\left(\begin{array}{ccc|c} 2 & 1 & 2 & 0.13-0.03 \\ 1 & 3 & 2 & 0.23-0.03 \\ 2 & 2 & 4 & 0.33-0.03 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0.2 \\ 2 & 1 & 2 & 0.1 \\ 2 & 2 & 4 & 0.3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0.2 \\ 0 & -5 & -2 & -0.3 \\ 0 & -4 & 0 & -0.1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 3 & 2 & 0.2 \\ 0 & 1 & 0 & 0.025 \\ 0 & 5 & 2 & 0.3 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0.125 \\ 0 & 1 & 0 & 0.025 \\ 0 & 0 & 2 & 0.175 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -0.05 \\ 0 & 1 & 0 & 0.025 \\ 0 & 0 & 1 & 0.0875 \end{array}\right).$$

Therefore, $(v_1, v_2, v_3) = (-0.05, 0.025, 0.0875)$. After normalization (setting $w_i := v_i / \sum v_j$), we get $(w_1, w_2, w_3) = (-0.8, 0.4, 1.4)$.

8 Appendix

Sample R code for generating the feasible set for 3 assets. (If this is any typo, do let me know.)

```
# Plotting the feasible set of 3 assets
# mean of the 3 assets
mA = 0.2
mB = 0.25
mC = 0.3

# sigma of the 3 assets
sA = 0.2
sB = 0.3
sC = 0.35

# correlation between the 3 assets
rhoAC = 0.5
rhoAB = 0.7
rhoBC = 0.2

# generating 3 numbers that sum to 1
n = 20000
x = matrix(runif(2*n),nc=2)
m = apply(x,1,min)
```

```
M = apply(x,1,max)
u1 = m
u2 = M-m
u3 = 1-M

# Feasible set
mP = u1*mA + u2*mB + u3*mC
vP = u1**2*sA**2 + u2**2*sB**2 + u3**2*sC**2 + 2*u1*u3*rhoAC*sA*sC +
2*u2*u3*rhoBC*sB*sC + 2*u1*u2*rhoAB*sA*sB
sP = sqrt(vP)
plot(sP,mP,col="grey",main="Feasible Set (no short selling)
(a)",xlab=expression(sigma),ylab="")
mtext(expression(mu), side=2, line=3)

# connecting A,C
a = seq(0,1,0.001)
mac = a*mA + (1-a)*mC
vac = a**2*sA**2 + (1-a)**2*sC**2 + 2*a*(1-a)*sA*sC*rhoAC
sac = sqrt(vac)
points(sac,mac, col=2,type="p")

# connecting A, B
mab = a*mA + (1-a)*mB
vab = a**2*sA**2 + (1-a)**2*sB**2 + 2*a*(1-a)*sA*sB*rhoAB
sab = sqrt(vab)
points(sab,mab, col=3,type="p")

# connecting B, C
mbc = a*mB + (1-a)*mC
vbc = a**2*sB**2 + (1-a)**2*sC**2  + 2*a*(1-a)*sB*sC*rhoBC
sbc = sqrt(vbc)
points(sbc,mbc, col=4)
```